algebra, geometry, and Schroedinger atoms

PILJIN YI
Korea Institute for Advanced Study

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Atiyah-Singer Index Theorem ~ 1963
Calabi-Yau ~ 1978
Calibrated Geometry ~ 1982
(Harvey & Lawson)

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Homological Mirror Symmetry ~ 1994
(Kontsevich)

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Stability & Derived Category ~ 2000

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Wall-Crossing Conjecture~ 2008
(Conjecture by Kontsevich & Soibelman)

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1975 ~ Bogomolnyi-Prasad-Sommerfeld (BPS)
1977 ~ Supersymmetry
1983 ~ Superstring Theory

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1985 ~ Calabi-Yau Compactification
1988 ~ Mirror Symmetry

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1994 ~ Wall-Crossing Discovered
(Seiberg & Witten)
1995 ~ Dirichlet Branes
1998 ~ Wall-Crossing is Bound State
Dissociation (Lee & P.Y.)

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2001 ~ Wall-Crossing for Black Holes
(Denef)

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2008 ~ Konstevich-Soibelman Explained
(Gaiotto & Moore & Neitzke)

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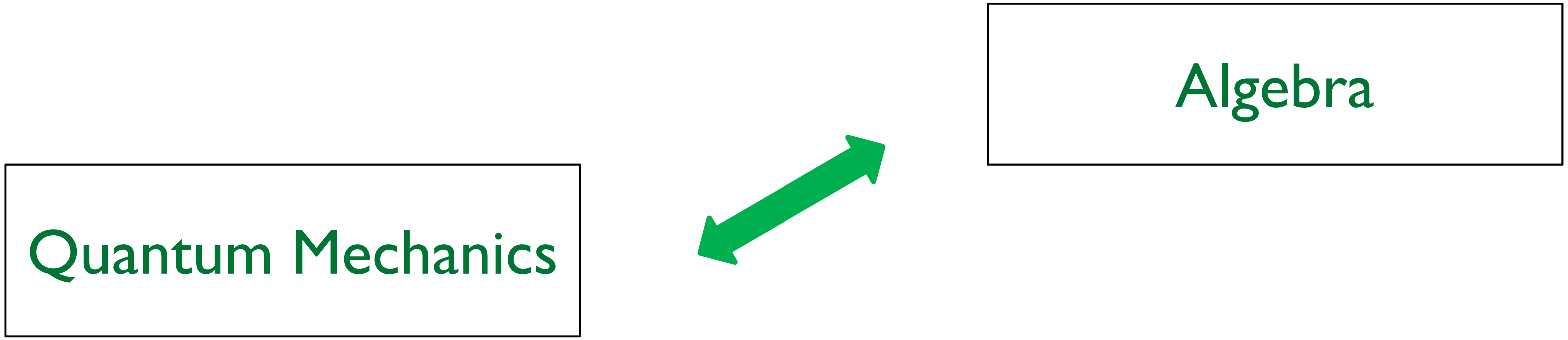
2011 ~ KS Wall-Crossing proved
via Quatum Mechanics
Manschot , Pioline & Sen /
Kim , Park, Wang & P.Y. / Sen

quantum is algebraic

spacetime is geometric

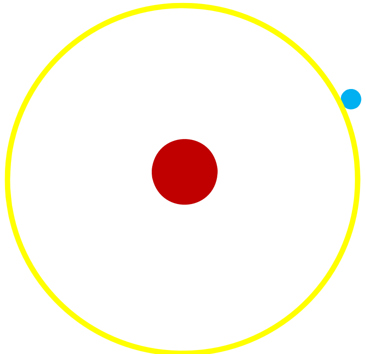
a little bit of superstring theory

how quantum mechanics solved a modern geometry problem

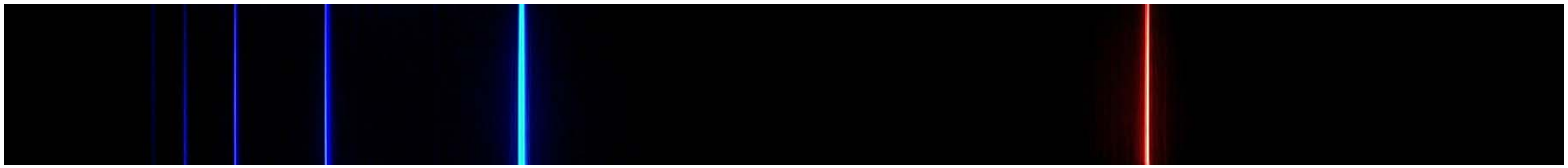

Algebra
Quantum Mechanics

quantum is algebraic

Balmer / Rydberg ~ 1880's

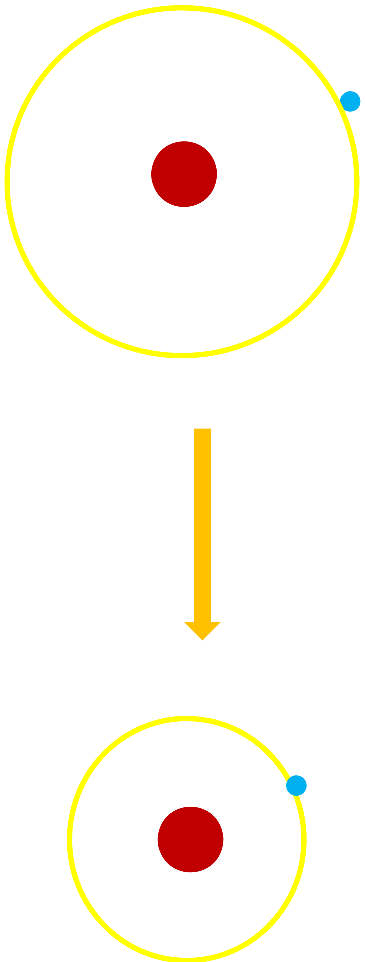


Hydrogen Atom



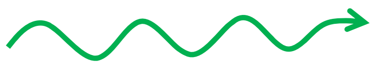
$$\lambda \sim 10^{-4} cm - 10^{-5} cm \quad = 10000\text{Å} - 1000\text{Å}$$

Balmer / Rydberg



$$\text{Hydrogen Atom}\Big|_{k}$$

$$n, k = 1, 2, 3, \ldots$$

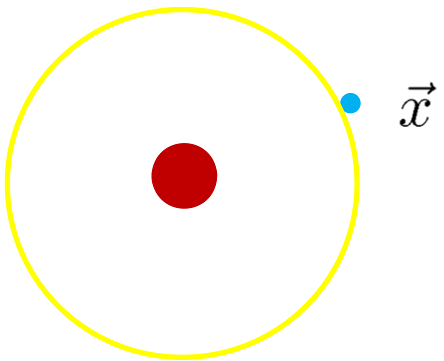


$$\Delta E = E_0 \times \left(\frac{1}{n^2} - \frac{1}{k^2} \right)$$

$$= E_k - E_n$$

$$\text{Hydrogen Atom}\Big|_{n}$$

Schroedinger Atoms



$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

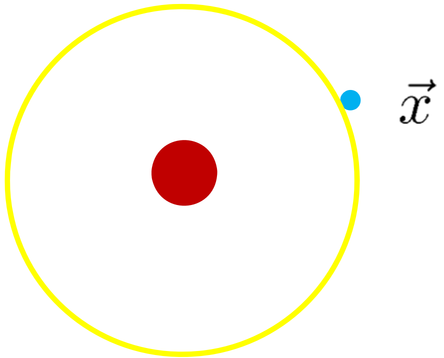
$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

$$[p_1, x^1] \equiv p_1 x^1 - x^1 p_1 = i\hbar$$

$$[p_2, x^2] = i\hbar$$

$$[p_3, x^3] = i\hbar$$

Schroedinger Atoms



$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

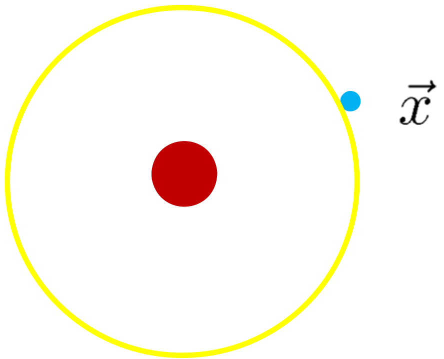
$$\vec{L} = \vec{x} \times \vec{p}$$

$$[H,\vec{L}] = 0 = [H,\vec{K}] \quad \Longrightarrow \quad SO(4) \sim \begin{pmatrix} 0 & \tilde{K}_1 & \tilde{K}_2 & \tilde{K}_3 \\ -\tilde{K}_1 & 0 & L_3 & -L_2 \\ -\tilde{K}_2 & -L_3 & 0 & L_1 \\ -\tilde{K}_3 & L_2 & -L_1 & 0 \end{pmatrix}$$

$$\vec{K} = \frac{1}{2m_e}\left(\vec{L}\times\vec{p} - \vec{p}\times\vec{L}\right) + e^2\frac{\vec{x}}{|\vec{x}|}$$

$$\tilde{K}_i = K_i\sqrt{\frac{m_e}{-2H}}$$

Schroedinger Atoms



$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

$E_1 = -E_0 \simeq -13.6\ eV$ $\qquad$ 2×1 distinct $|\Psi_{1,\ldots}\rangle$'s

$E_2 = -E_0/4 \simeq -3.4\ eV$ $\qquad$ 2×4 distinct $|\Psi_{2,\ldots}\rangle$'s

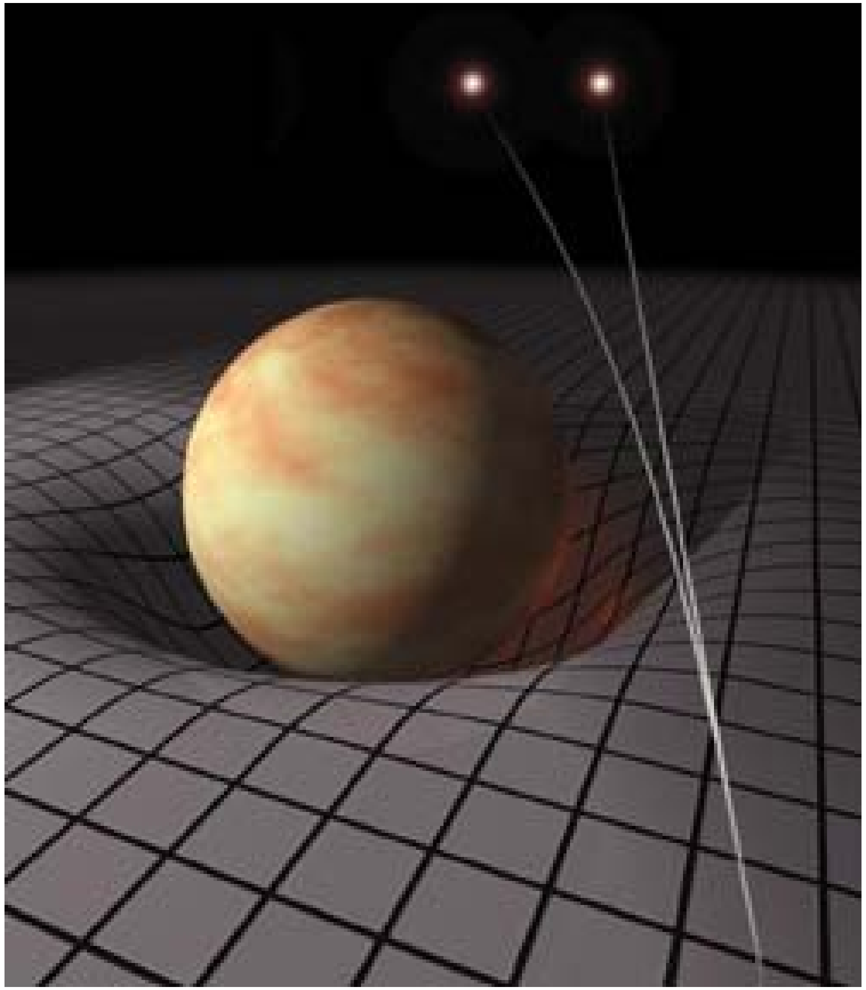
$E_3 = -E_0/9 \simeq -1.5\ eV$ $\qquad$ 2×9 distinct $|\Psi_{3,\ldots}\rangle$'s

General Relativity

Geometry

spacetime is geometric

Einstein ~ 1915

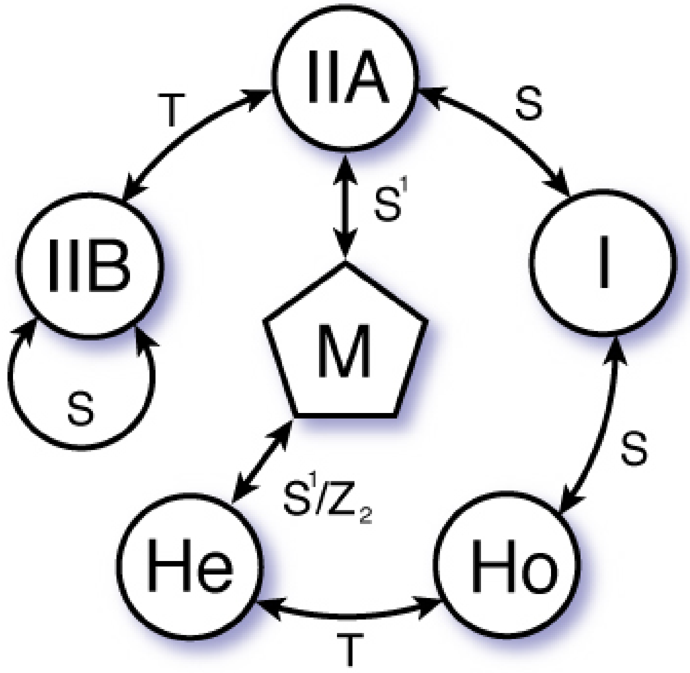


$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}$$

Kaluza & Klein (~1921) asked :

can the world has more than
3 spatial dimensions or 4 spacetime dimensions
with the extra directions curled up so small to be practically invisible
and, if so, what are the physical consequences ?

after all, space (& time) is supposed to curved five superstring theories live in 10 dimensional spacetime



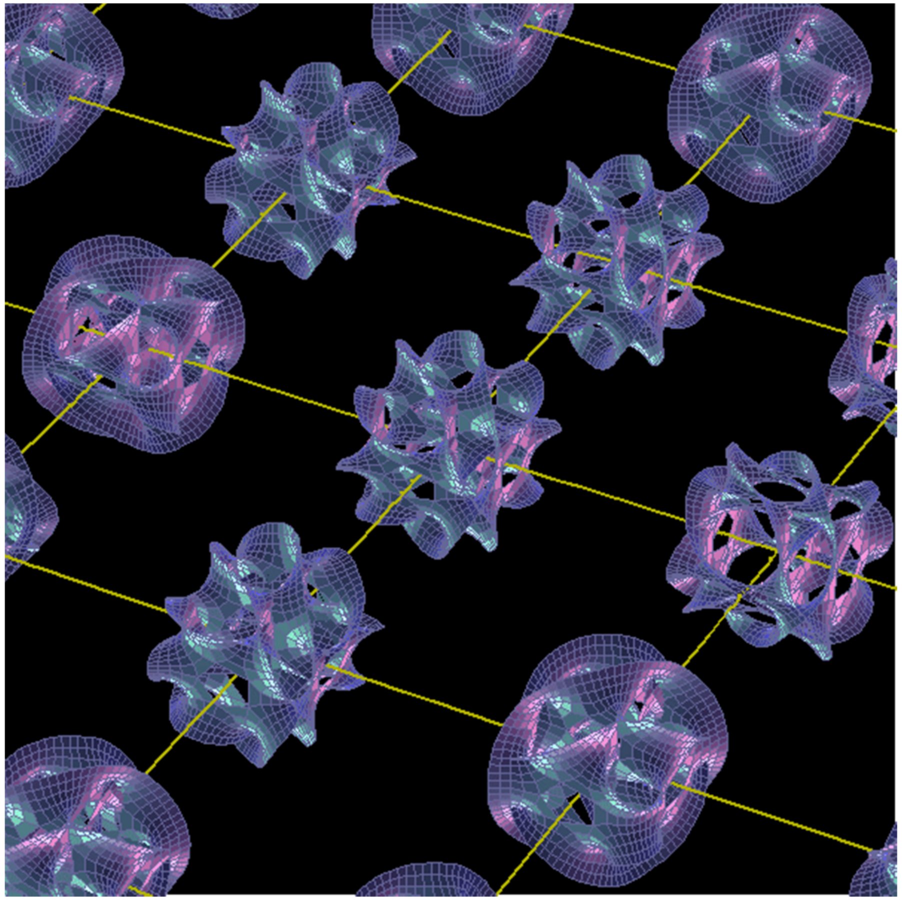
superstring theory says :

spacetime is composed of 4+6 dimensions with
very small & tightly-curved (say, Calabi-Yau) 6D manifold
sitting at each and every point of usual 3D space,

roughly,

or, more precisely,

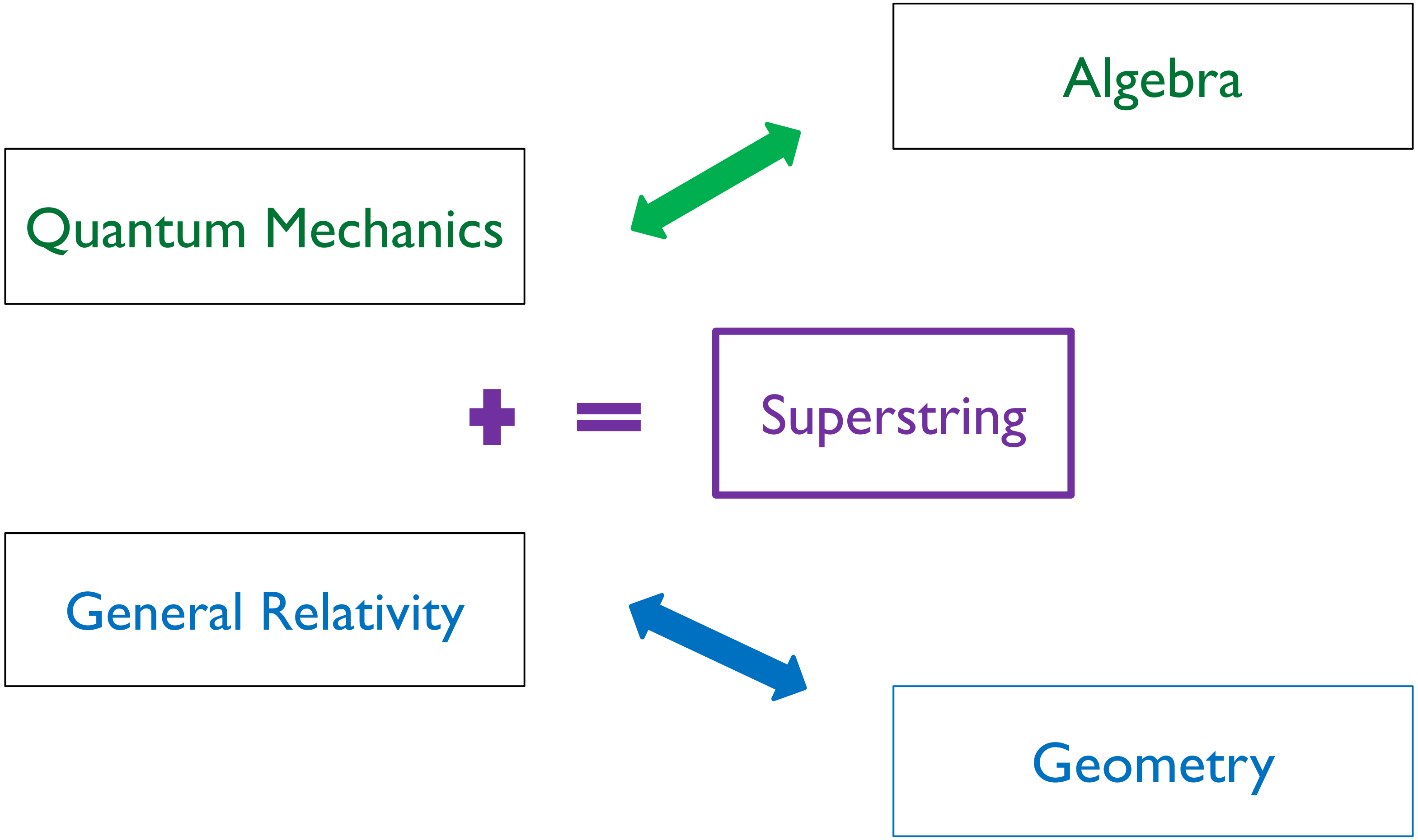
$$\longleftrightarrow \quad 10^{-13}cm \gg l_{size} > 10^{-33}cm$$



$$R_{AB} - \frac{1}{2}g_{AB}R = 8\pi\kappa_{9+1}^{2}T_{AB}$$

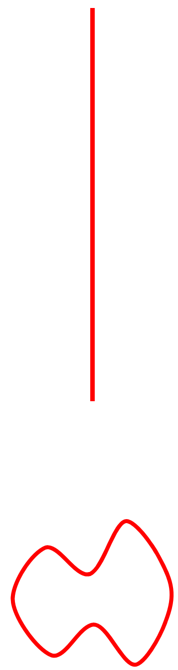
a little bit of superstring theory

particles from geometry / geometry from particles


Algebra
Quantum Mechanics
Superstring
General Relativity
Geometry

basic building blocks in superstring theory

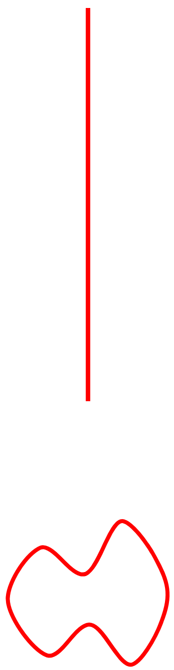
fundamental
strings



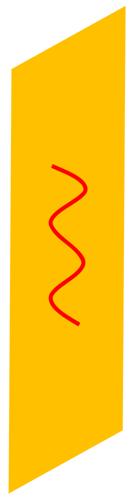
light particles
some of which mediate "forces"
such as gravity & electromagnetism

basic building blocks in superstring theory

fundamental
strings



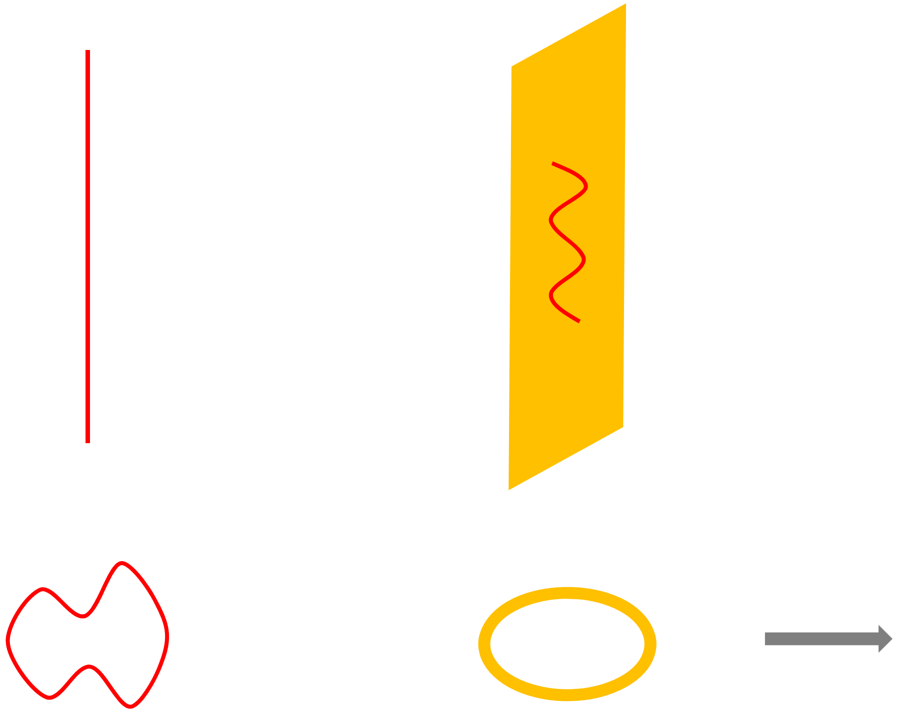
Dirichlet
branes
(as in membrane)



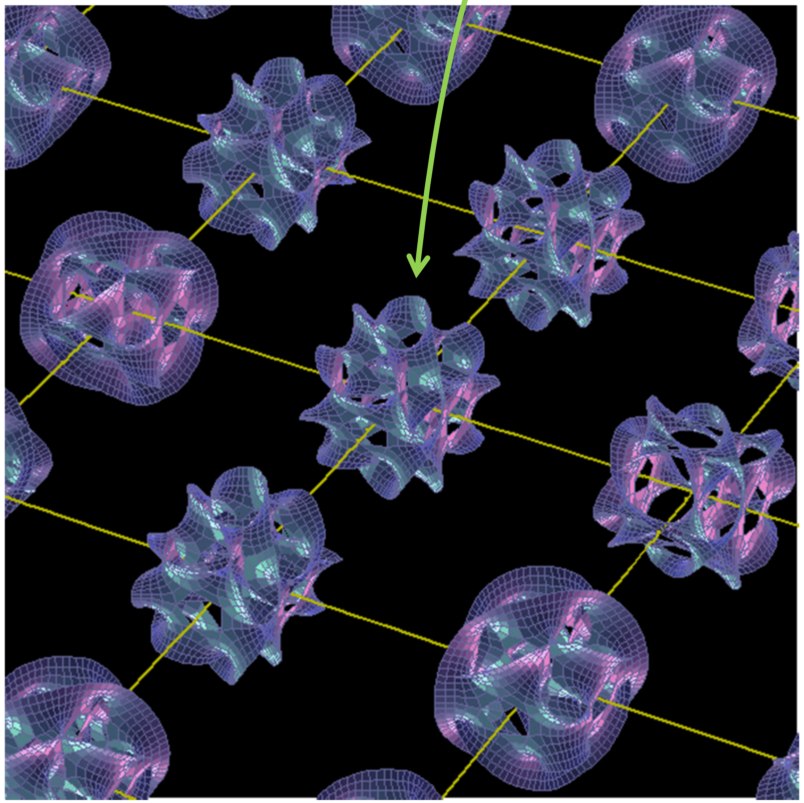
basic building blocks in superstring theory

fundamental strings

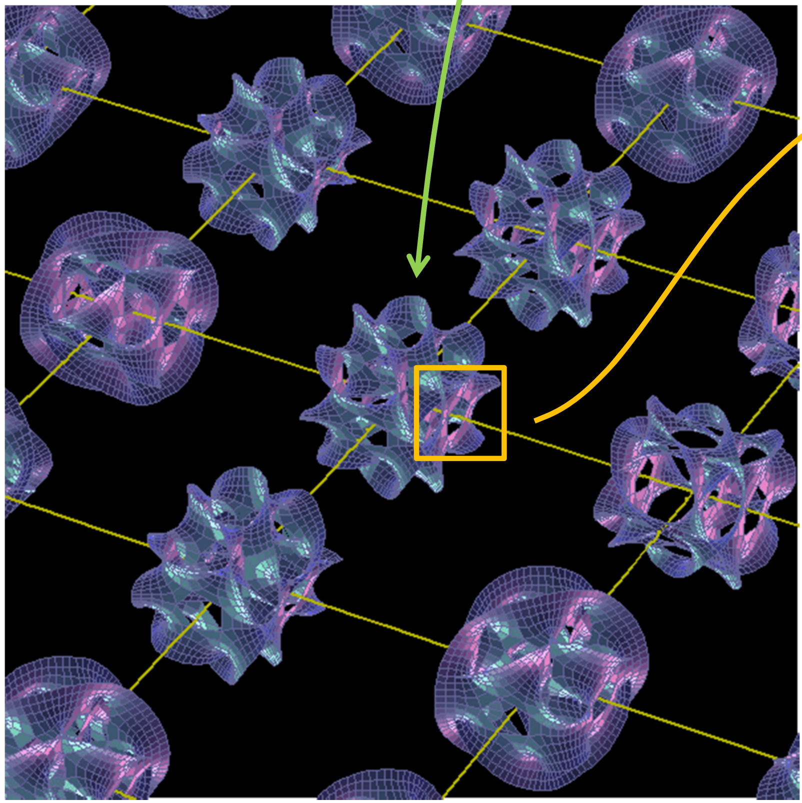
Dirichlet branes

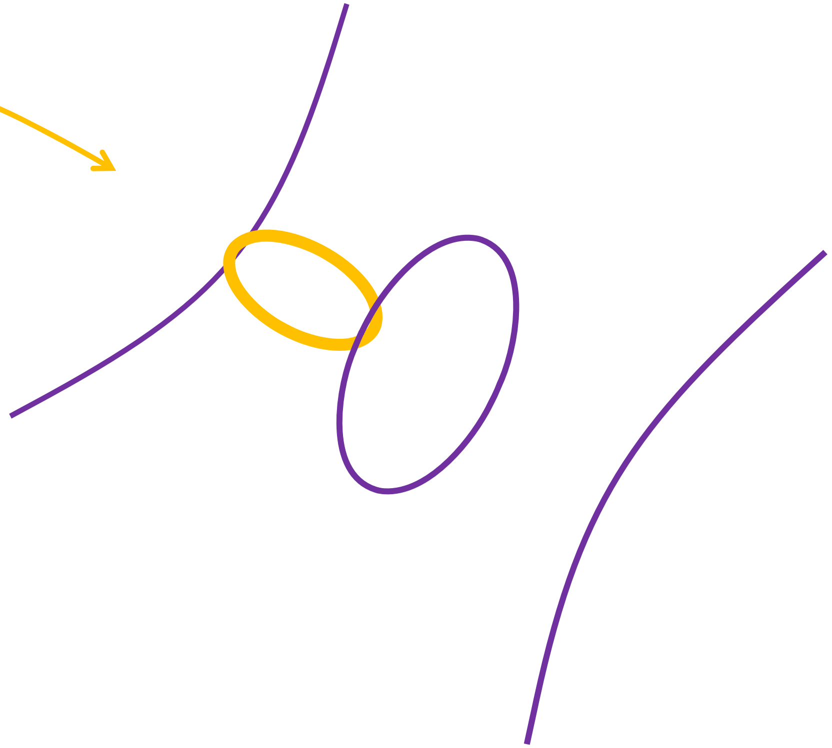


heavy particles, typically with electric/magnetic charges a particle located somewhere in our visible space

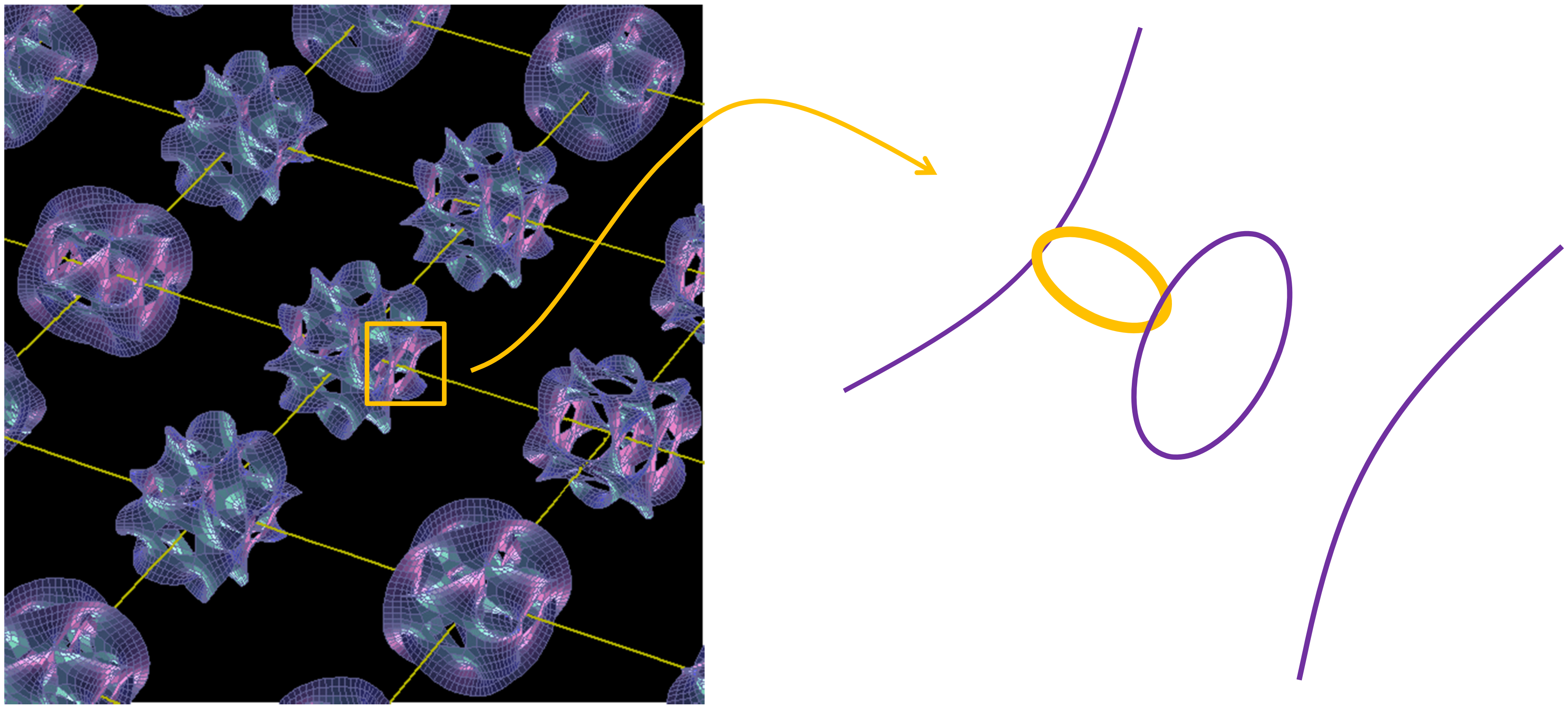


a particle located somewhere in our visible space
= a wrapped brane in the hidden Calabi-Yau at that point

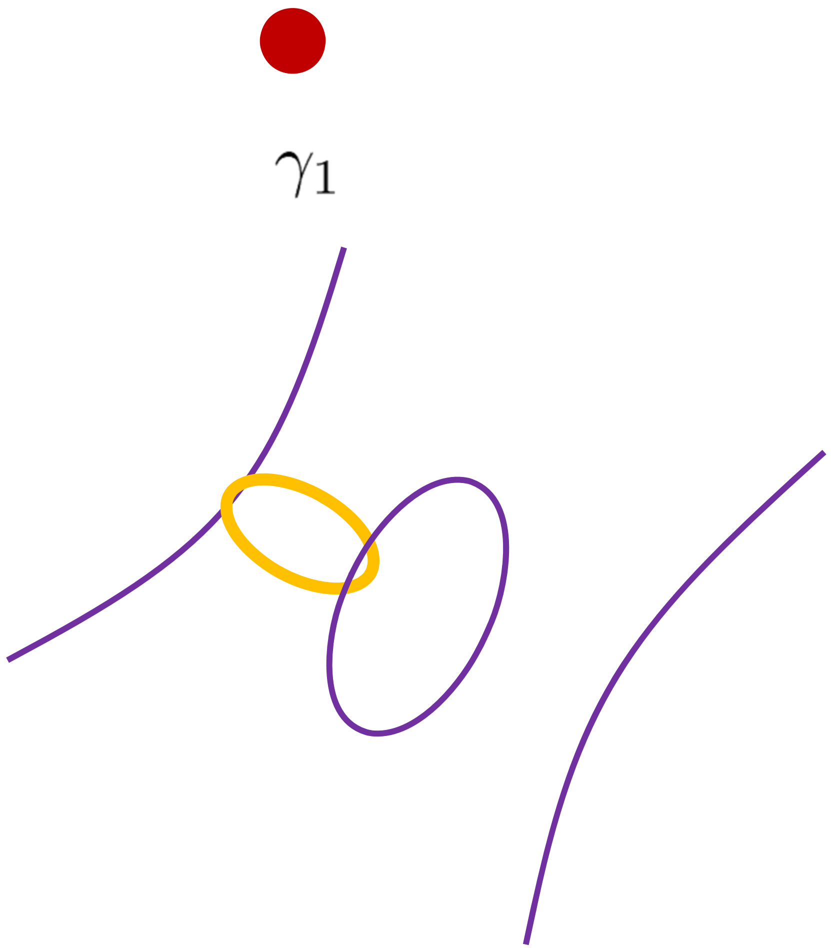




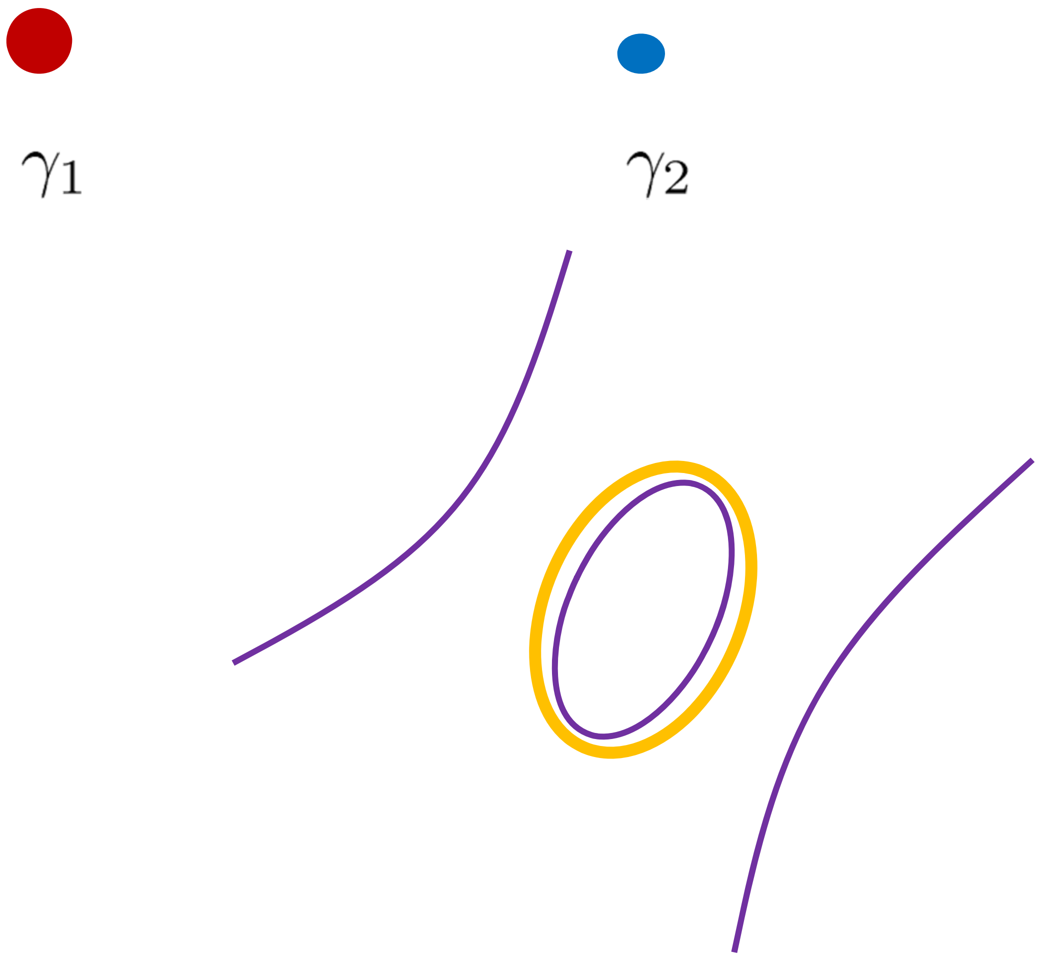
this means that we can actually detect
geometry (loops, holes, cavities, ……) of the hidden 6D space
by detecting what kind particles exist in visible 3D world



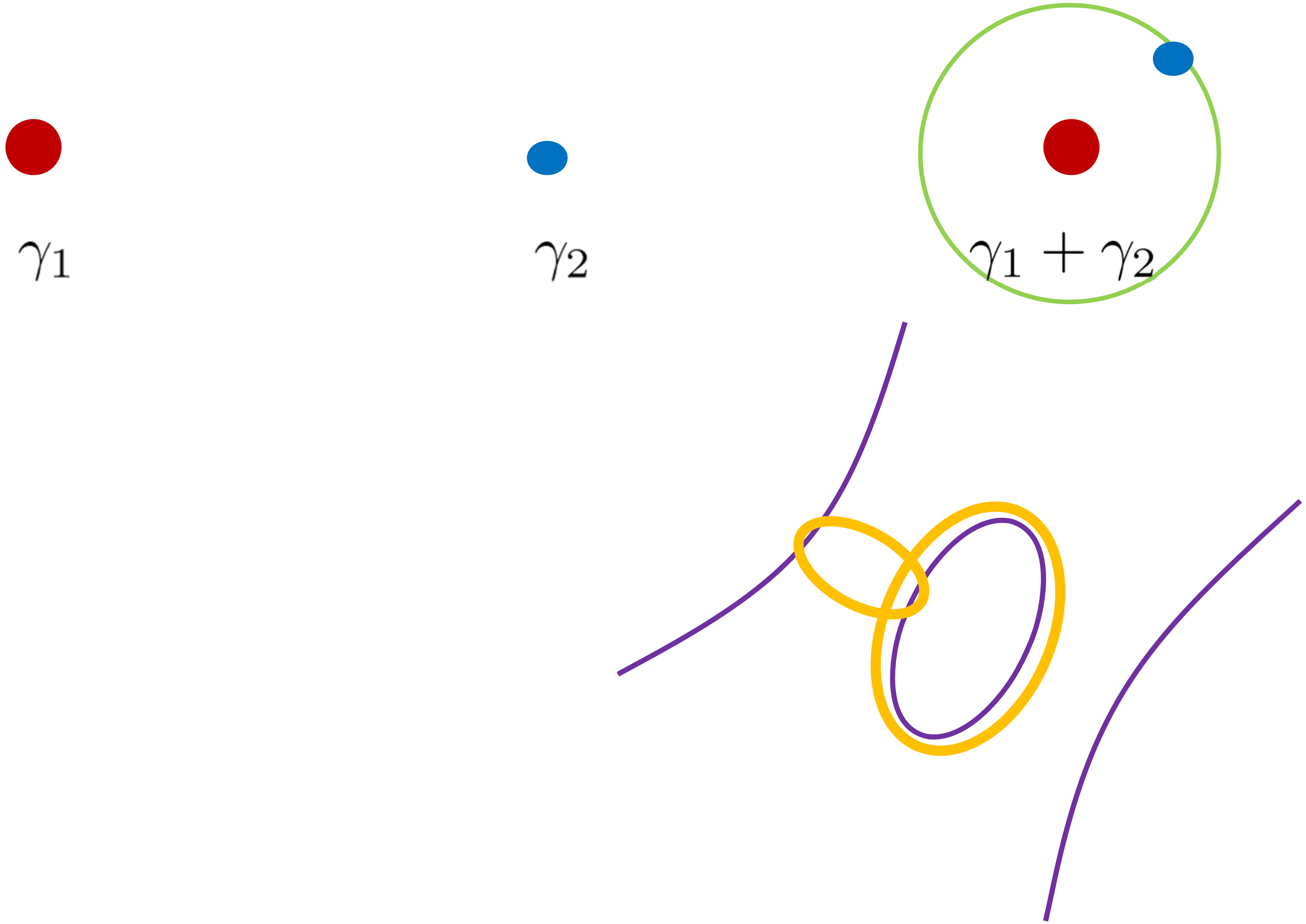
particle vs. geometry



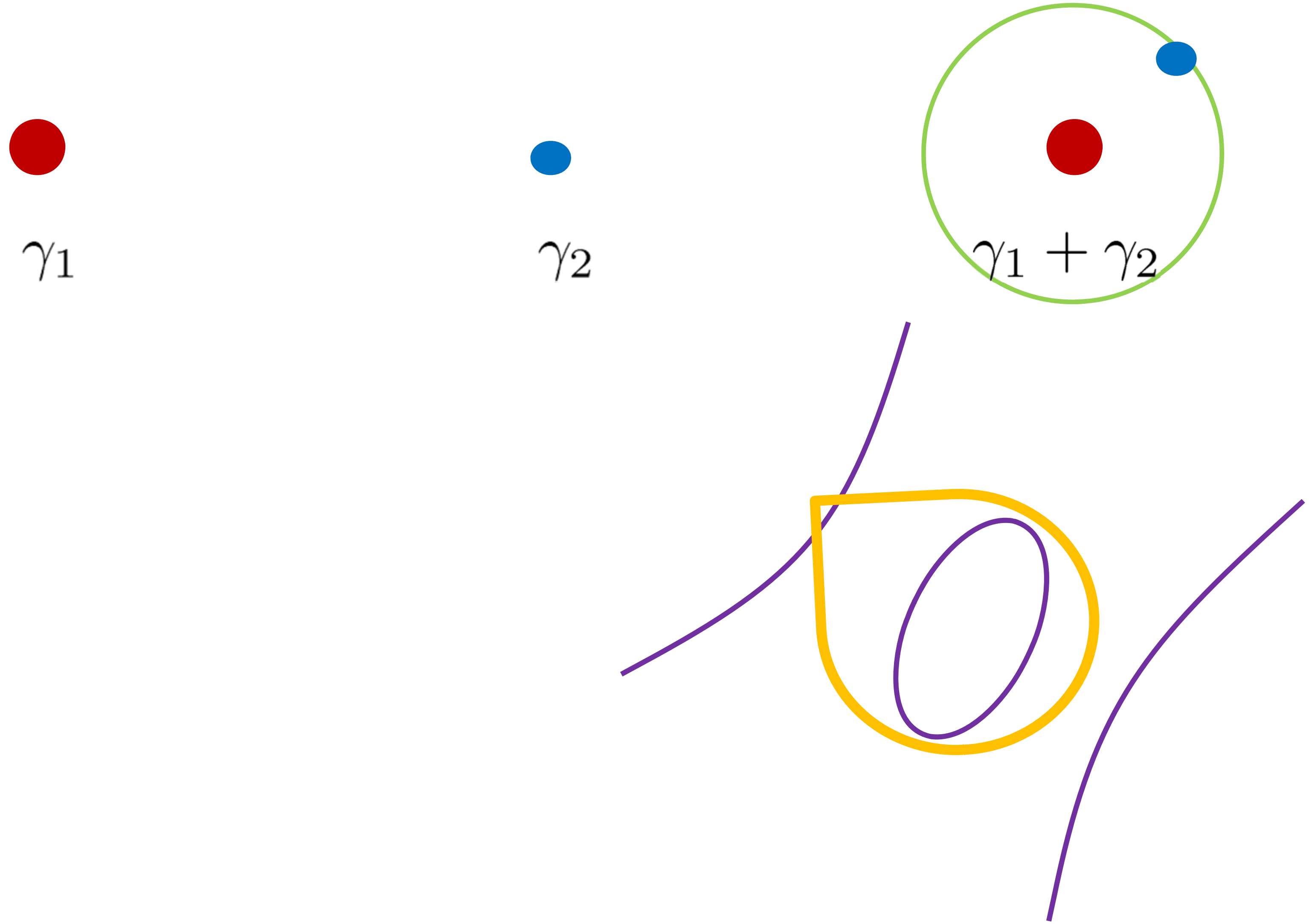
particle vs. geometry



particle vs. geometry

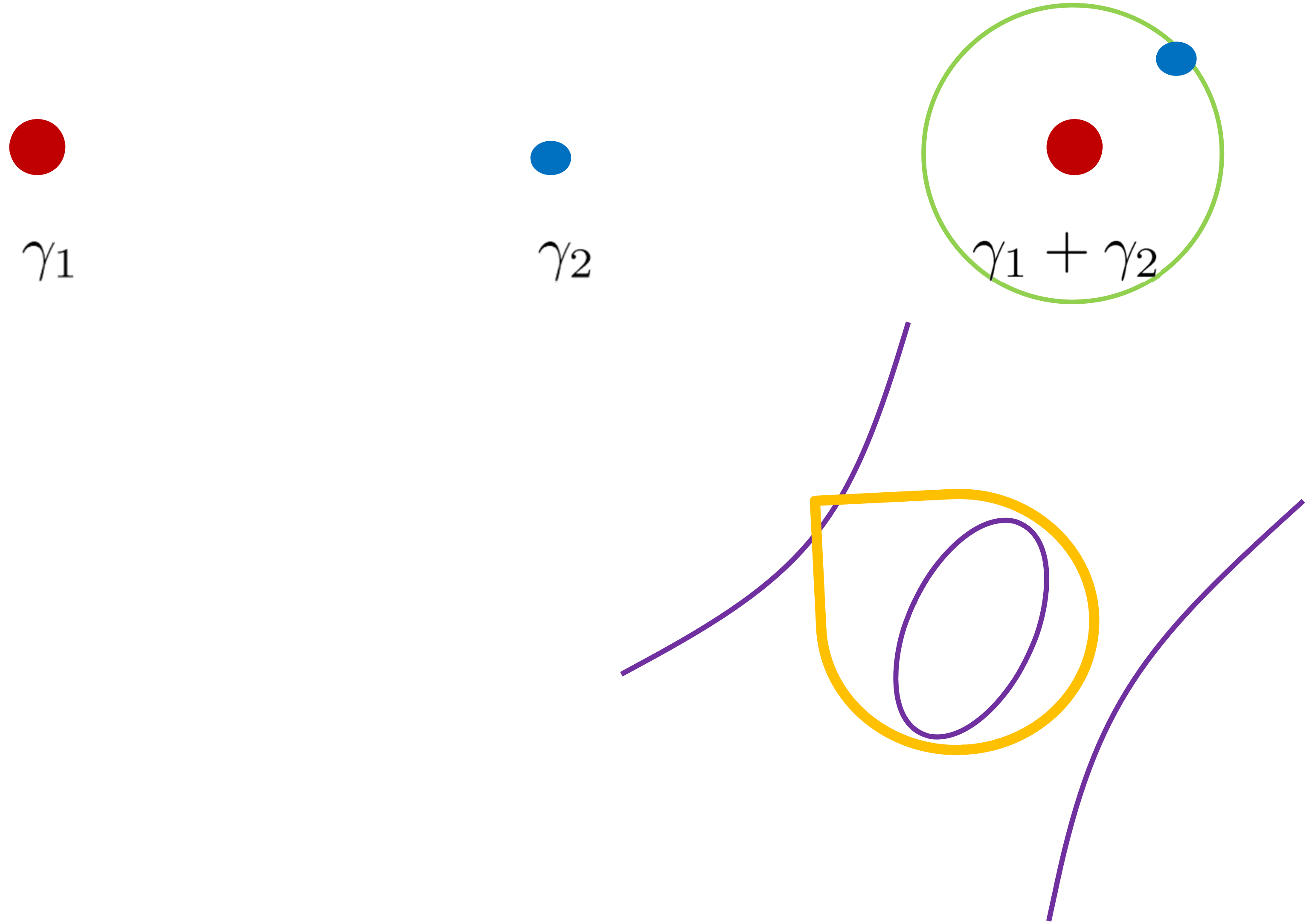


particle vs. geometry



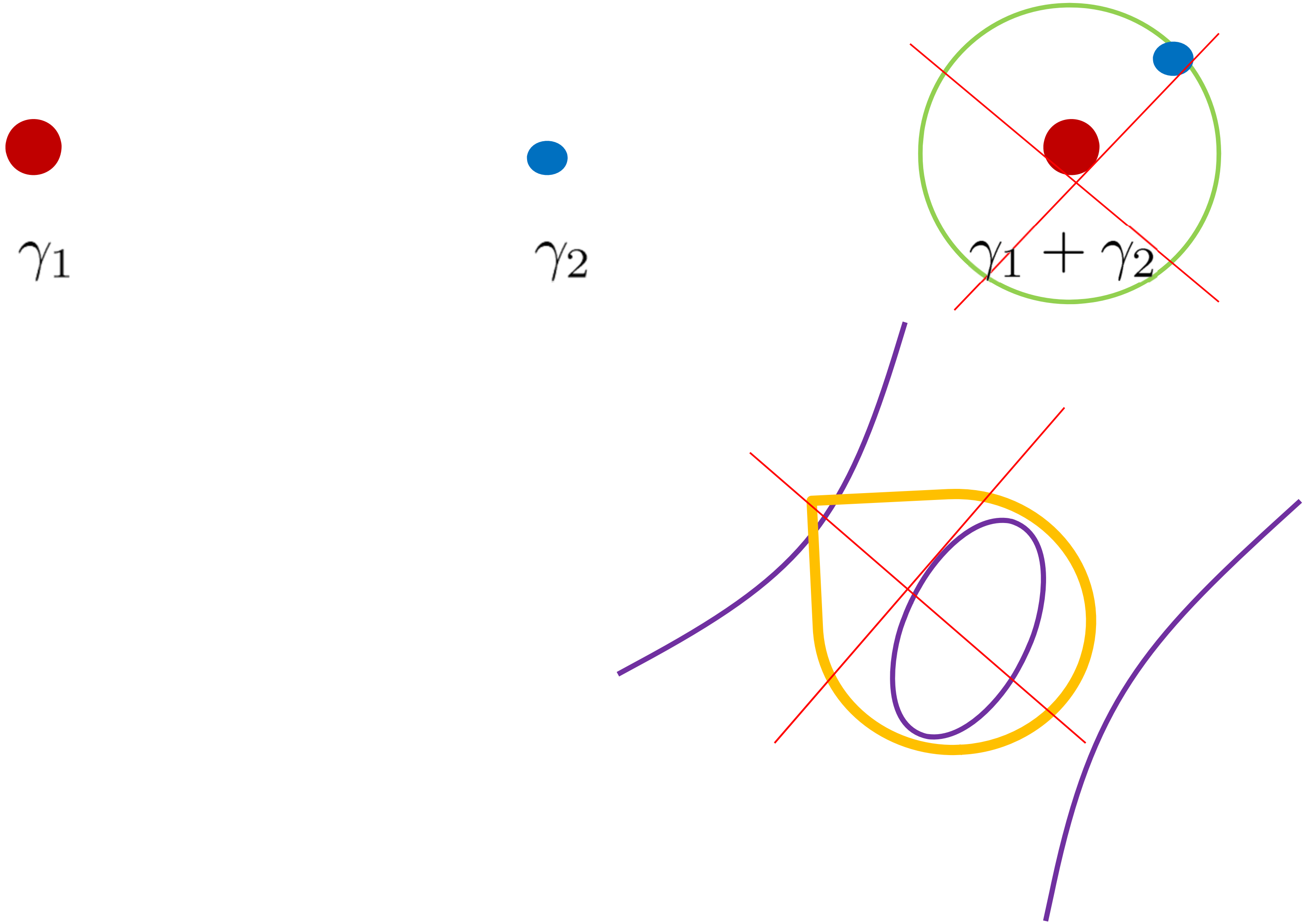
existence of "quantum BPS (bound) states"

= existence of "calibrated 3-cycles"



non-existence of "quantum BPS (bound) states"

= non-existence of "calibrated 3-cycles"

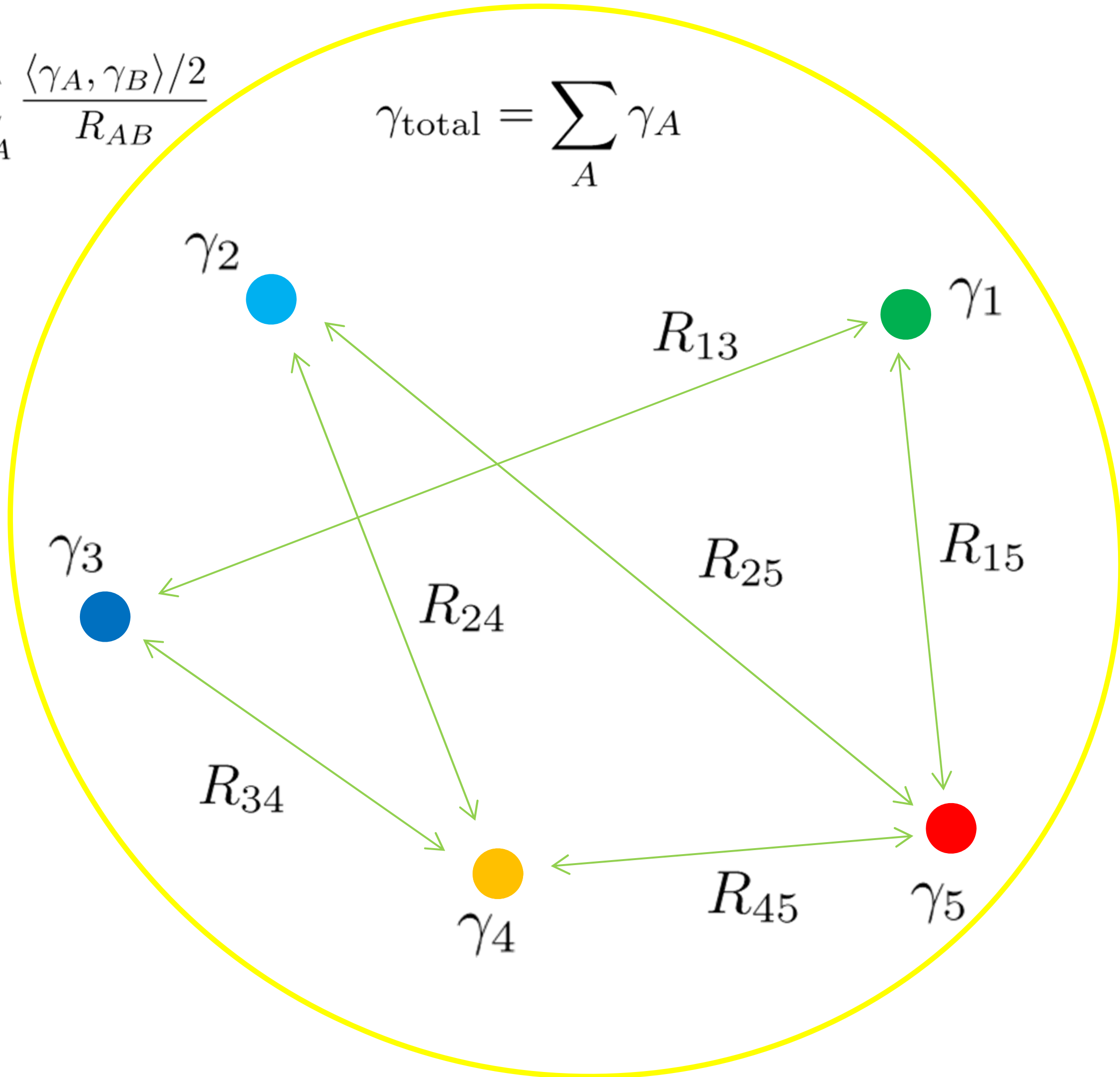


how quantum mechanics solved a modern geometry problem

generic BPS particle is a multi-center bound state

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$



thus, wall-crossing has a very simple and interpretation

in the particle / quantum mechanics viewpoint

as bound states becoming unbound

$$\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \end{array}$$

$$\begin{array}{cc} \gamma_1 & \gamma_1 + \gamma_2 \\ \gamma_2 & \gamma_2 + \gamma_3 \\ \gamma_3 & \gamma_1 + \gamma_3 \\ \vdots & \vdots \\ & \gamma_1 + \gamma_2 + \gamma_3 \\ & \vdots \end{array}$$

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}]\langle \gamma_A, \gamma_B \rangle < 0$$

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}]\langle \gamma_A, \gamma_B \rangle > 0$$

quantum mechanics for many such charged particles

Kim+Park+P.Y.+Wang 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

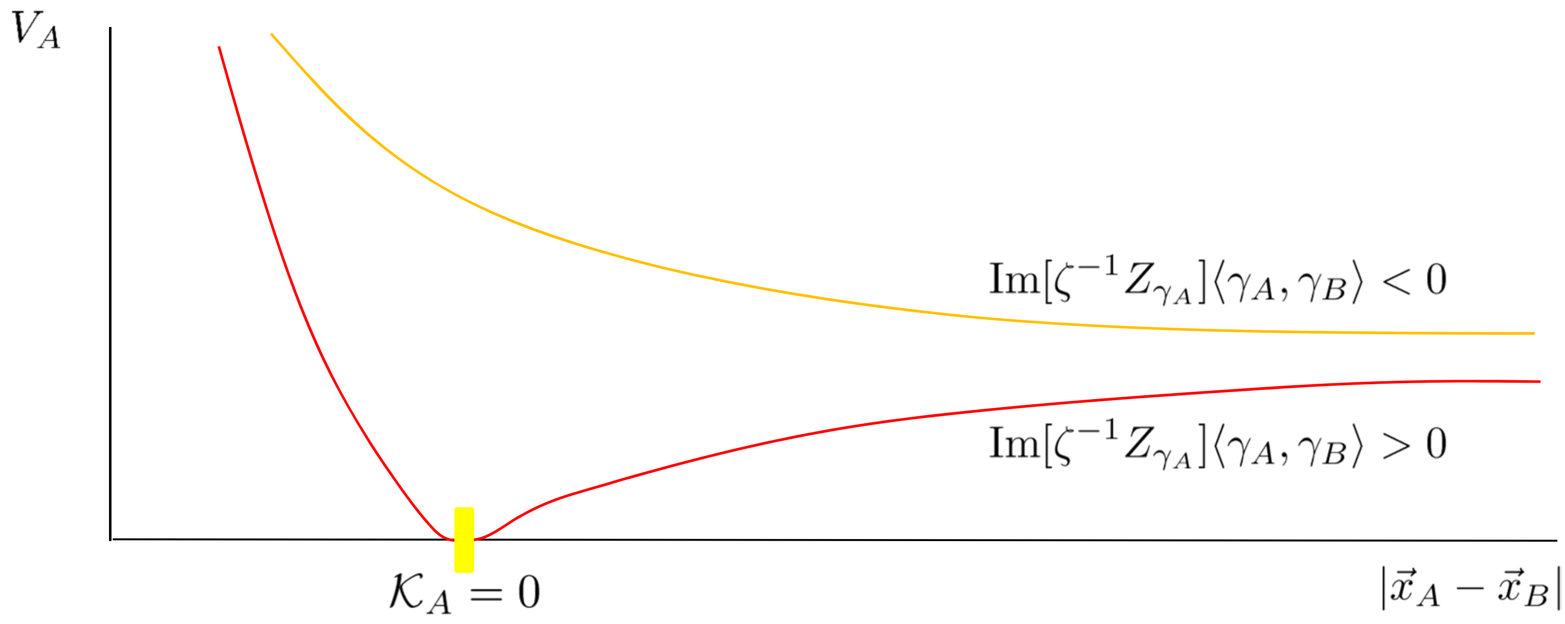
$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\ \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

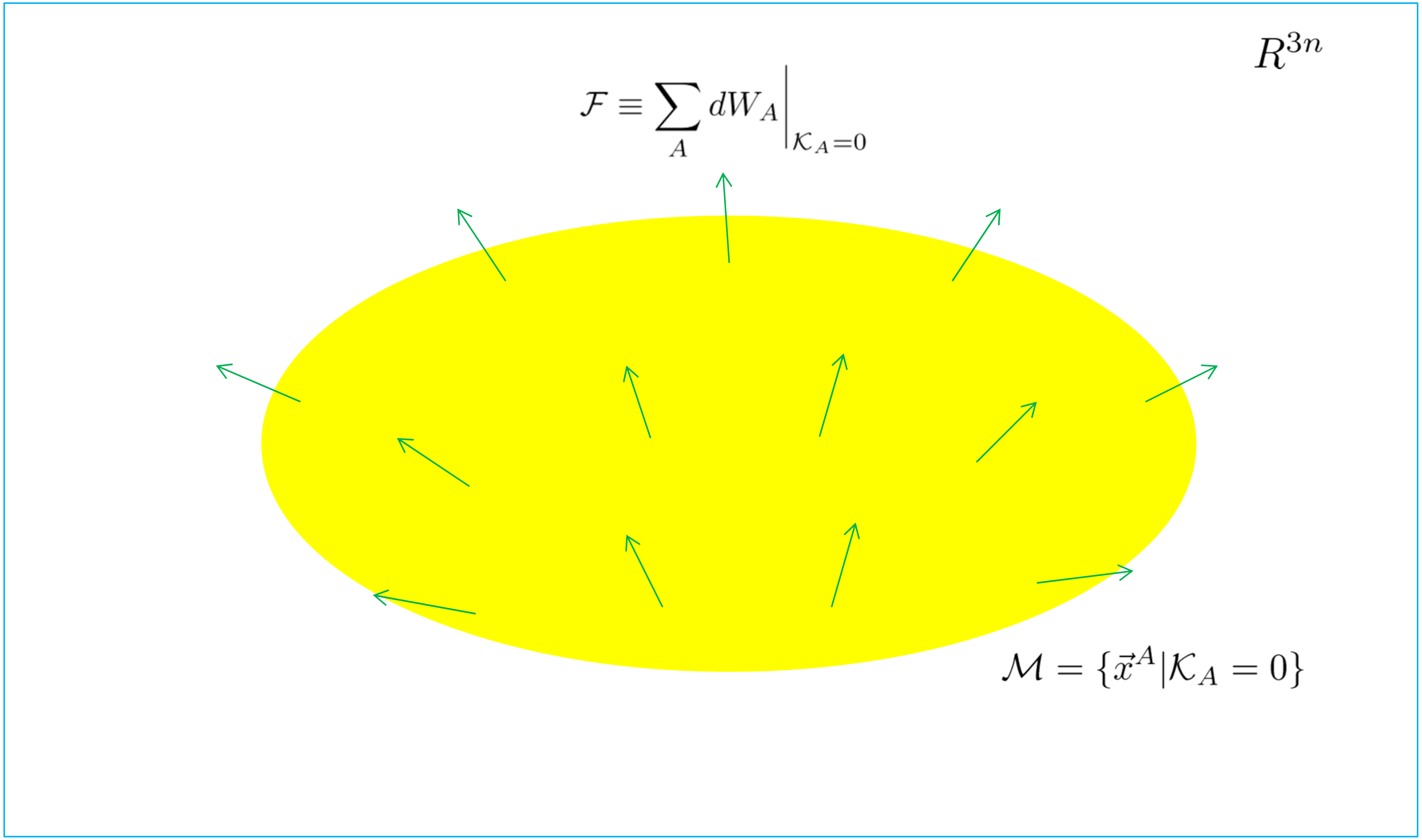
$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A |\vec{x}_A|^2 \qquad \text{asymptotically}$$

repulsive or attractive depending on a sign

$$V_A = \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_A])^2}{2|\mathcal{Z}_A|} \sim \left(\mathrm{Im}[\zeta^{-1}Z_{\gamma_A}] - \sum_B \frac{\langle\gamma_A,\gamma_B\rangle/2}{|\vec{x}_A - \vec{x}_B|}\right)^2$$



3n-dim dynamics → 3 + 2(n-1) dim nonlinear sigma model via deformation & localization that preserve the index



index theorem

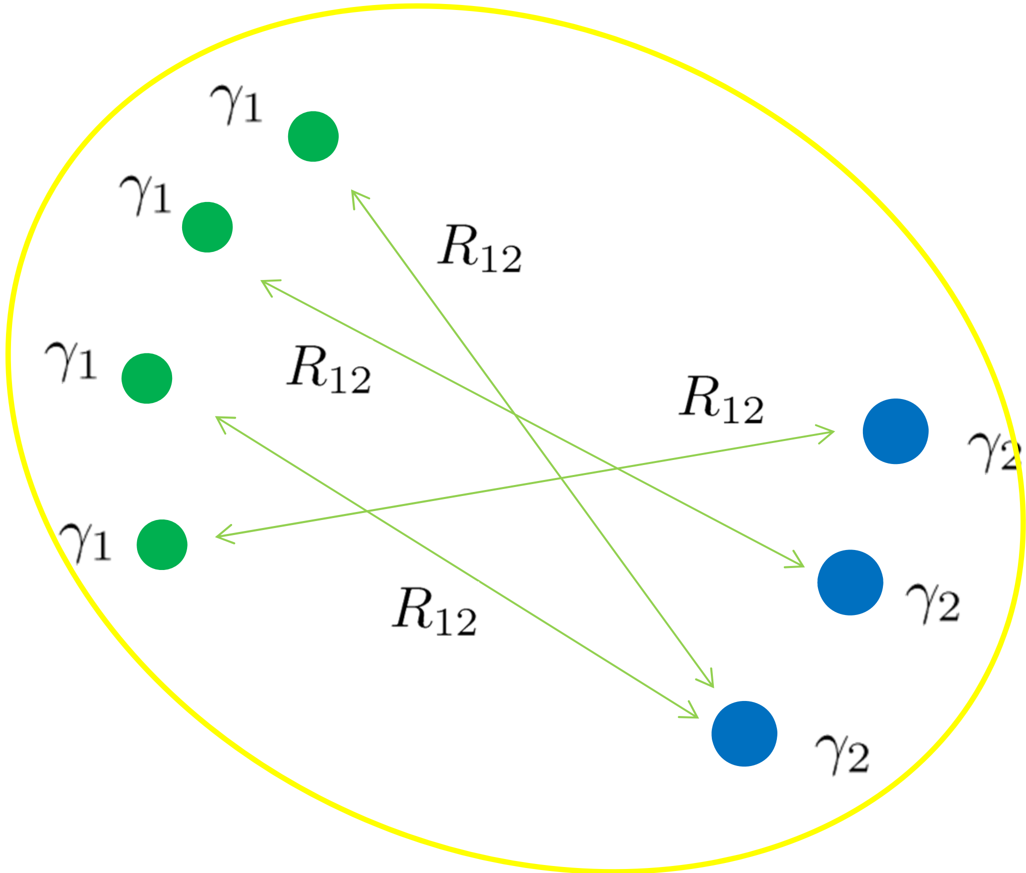
$$I_n(\{\gamma_A\}) = \mathrm{tr}\ \left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\ \left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A\ \mid\ \mathcal{K}_A=0\}} ch(\mathcal{F}) \wedge \mathbf{A}(\mathcal{M})$$

$$= \int_{\mathcal{M}=\{\vec{x}_A\ \mid\ \mathcal{K}_A=0\}} ch(\mathcal{F}) \wedge \det\left(\frac{R/2}{\sinh(R/2)}\right)$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle\gamma_A,\gamma_B\rangle}{2}\ \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

Bose/Fermi statistics is essential



universal wall-crossing formulae from quantum mechanics of BPS particles

Kim+Park+P.Y.+Wang 2011

$$\bar{\Omega}^-\left(\sum \gamma_A\right) - \bar{\Omega}^+\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle \gamma_A,\gamma_B\}+n-1}\frac{\prod_A \bar{\Omega}^+(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}} ch(\mathcal{F})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle \gamma'_{A'},\gamma'_{B'}\}+n'-1}\frac{\prod_{A'} \bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'} ch(\mathcal{F}')$$

$$\vdots$$

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2$$

$$+(-1)^{\sum_{A''>B''}\langle \gamma''_{A''},\gamma''_{B''}\}+n''-1}\frac{\prod_{A''} \bar{\Omega}^+(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''} ch(\mathcal{F}'')$$

$$\vdots$$

$$\sum_{A=1}^{n}\gamma_A = \cdots = \sum_{A'=1}^{n'}\gamma'_{A'} = \cdots = \sum_{A''=1}^{n''}\gamma''_{A''} = \cdots$$

2000 Stern + P.Y.

wall-crossing formula for simple magnetic charges; weak coupling regime

2002 Denef

quiver dynamics representation of N=2 supergravity BH's

⋮

2008 Kontsevich + Soibelman

⋮

2010/2011 Manschot + Pioline + Sen

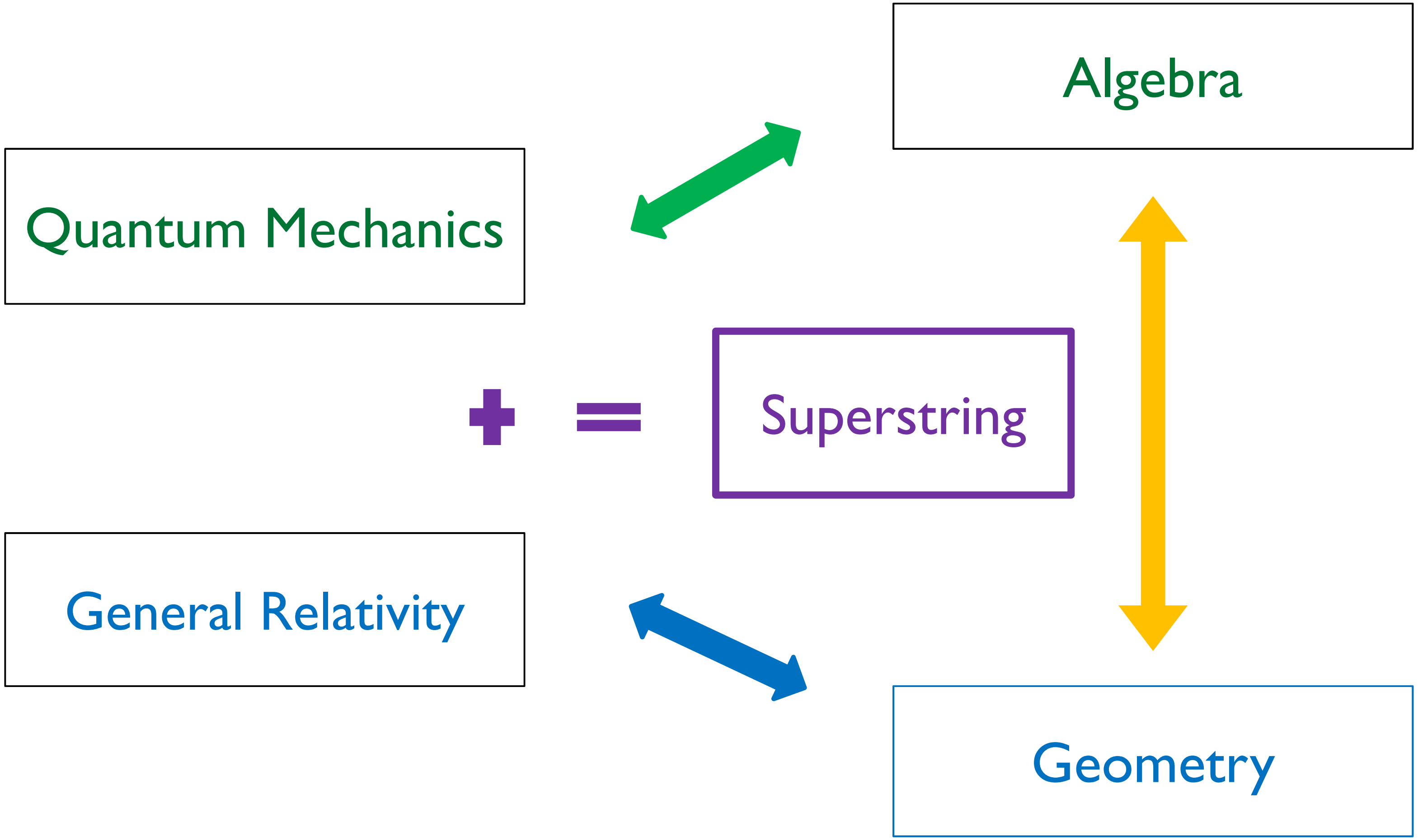
general n-particle conjecture for Quantum Mechanics Counting

2011 Lee+P.Y. / Kim+Park+Wang+P.Y.

general n-particle solution to Quantum Mechanics Counting

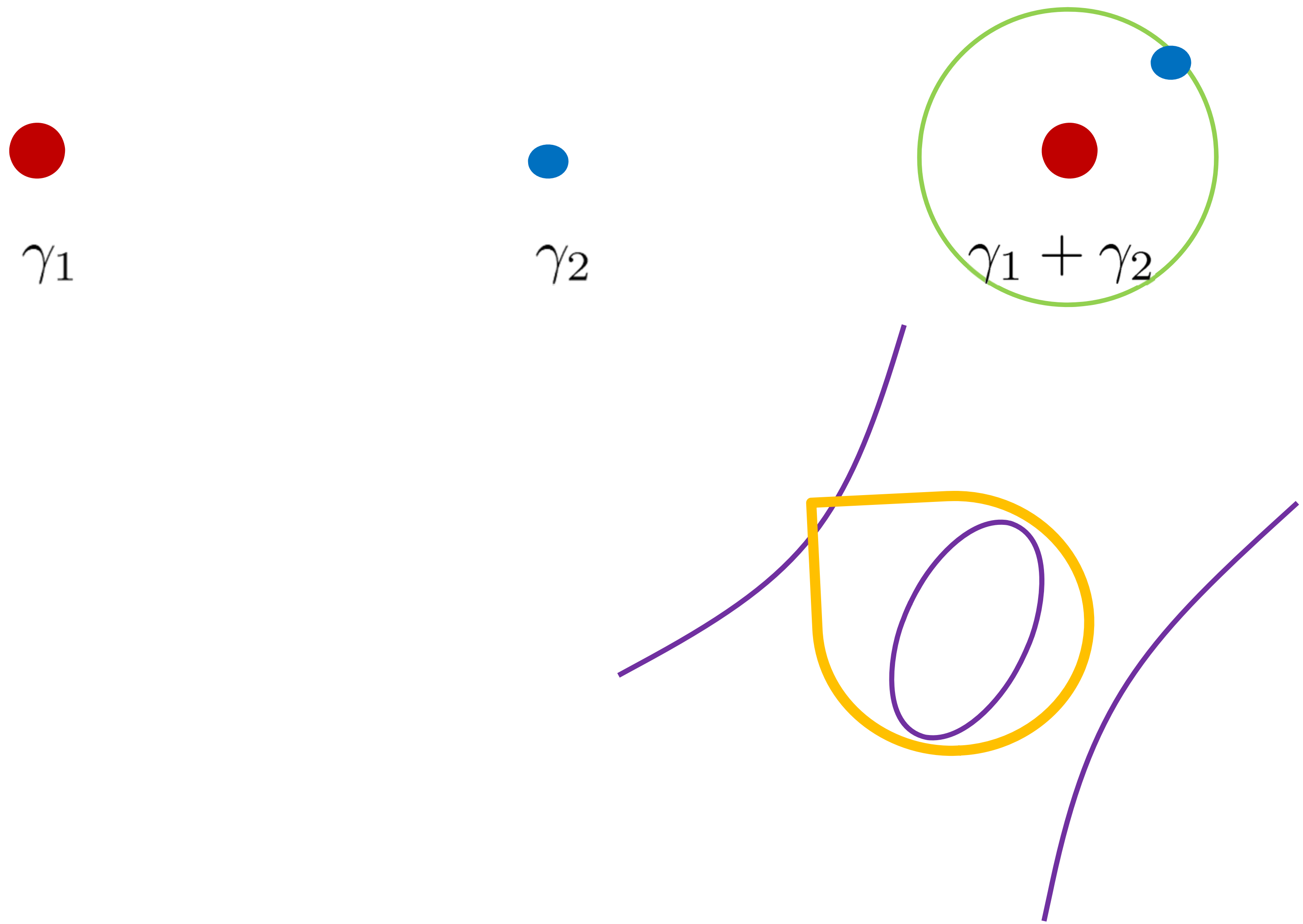
2011 Sen

Quantum Mechanics Counting = Kontsevich-Soibelman Conjecture


Algebra
Quantum Mechanics
+ =
Superstring
General Relativity
Geometry

existence of “quantum BPS (bound) states”

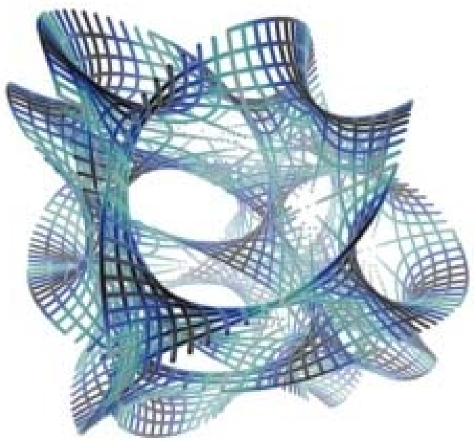
= existence of “calibrated 3-cycles”

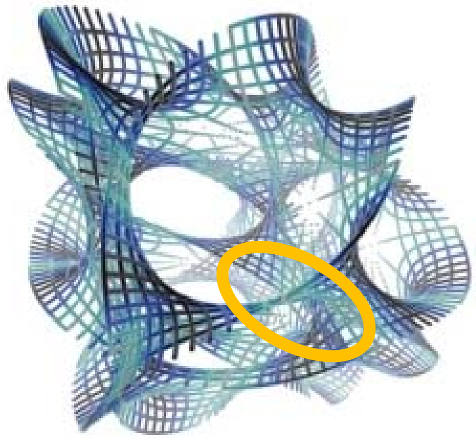


Calabi-Yau manifold & calibrated 3-cycles

$J^{(1,1)}$

$\Omega^{(3,0)}$

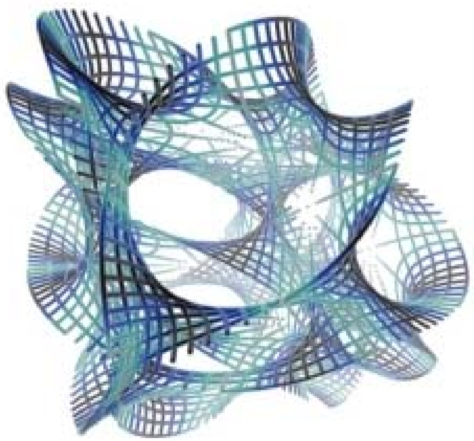


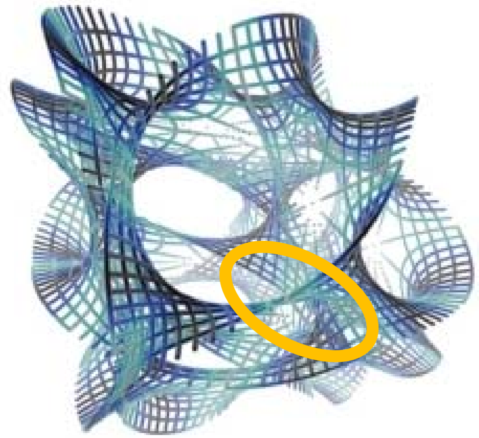


given a family of Calabi-Yau manifold with a fixed topology,
which topological 3-cycles can be calibrated ?

$$J^{(1,1)} \qquad\qquad J^{(1,1)}\Big|_{\bigcirc} = 0$$

$$\Omega^{(3,0)} \qquad\qquad e^{-i\alpha}\Omega^{(3,0)}\Big|_{\bigcirc} = \text{volume density of } \bigcirc$$



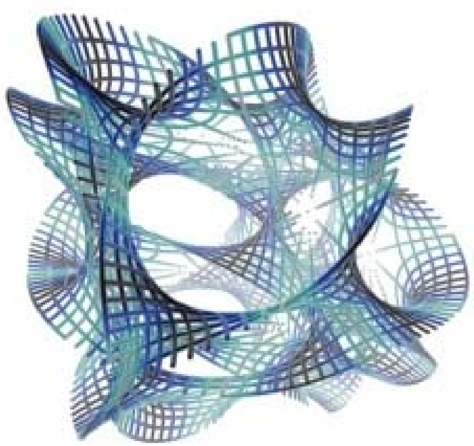


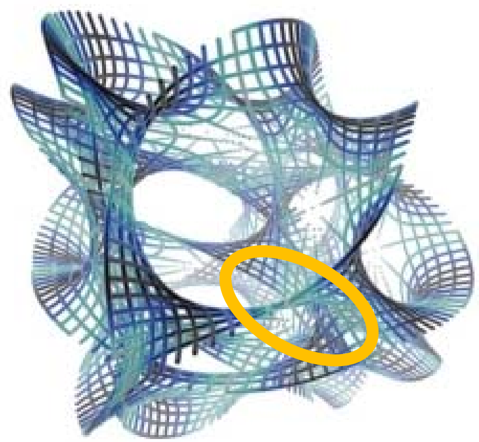
given a family of Calabi-Yau manifold with a fixed topology,
which topological 3-cycles can be calibrated ?

minimal volume condition
+
an extra "minimization" condition

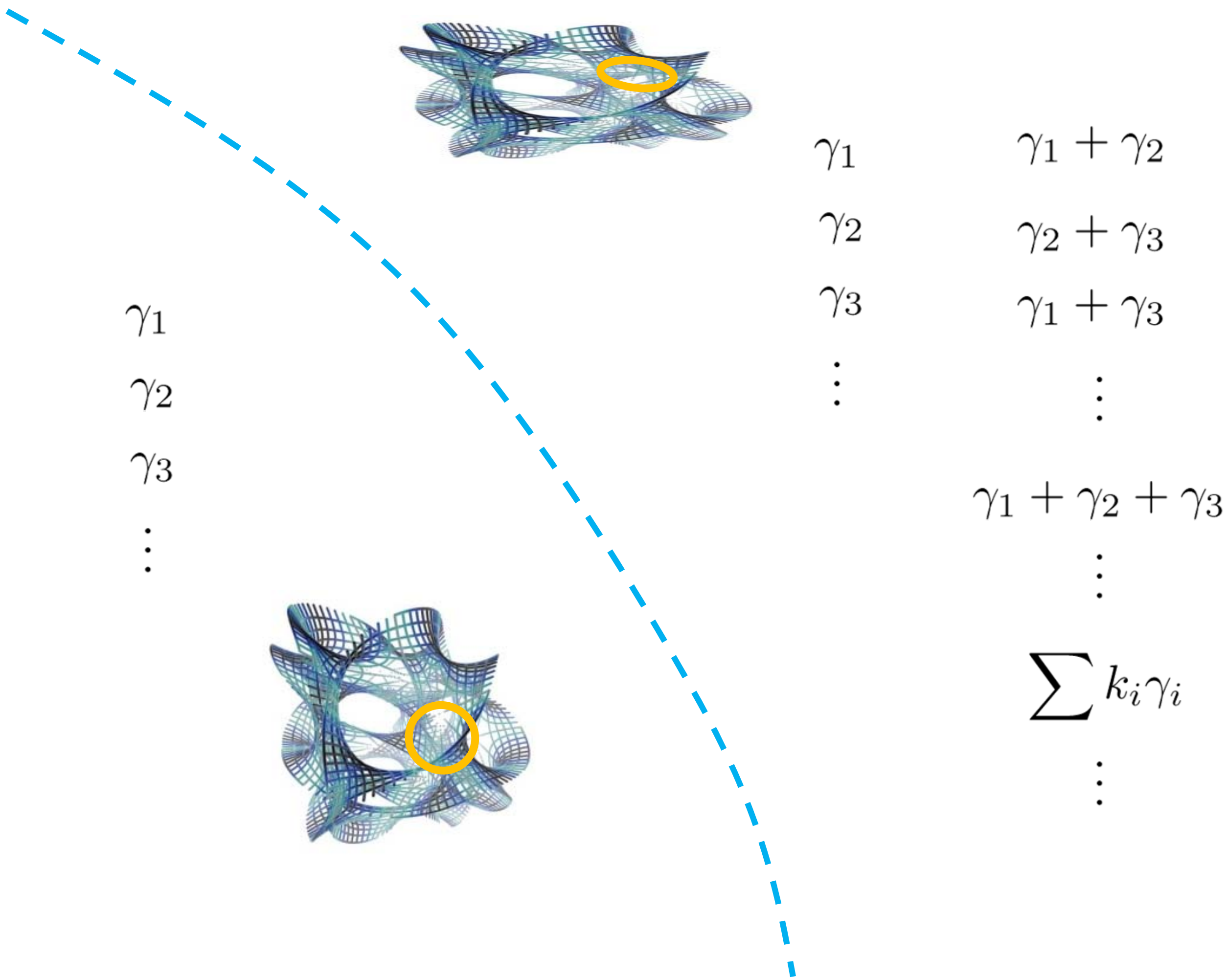
$$J^{(1,1)}\Big| = 0$$

$$e^{-i\alpha}\Omega^{(3,0)}\Big| = \text{volume density of}$$





which is the reason why topology alone cannot guarantee existence of such cycles



Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

Kontsevich-Soibelman conjecture : a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$\langle \gamma, \gamma' \rangle = \langle (g, e), (g', e') \rangle = g \cdot e' - e \cdot g' \ \in \ \mathbf{Z}$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$\downarrow$$

$$V_\gamma$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$\boxed{V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle V_{\gamma+\gamma'} \qquad \langle \gamma, \gamma' \rangle = 0, \pm 1, \pm 2, \ldots}$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle V_{\gamma+\gamma'} \qquad \langle \gamma, \gamma' \rangle = 0, \pm 1, \pm 2, \ldots$$

$$\Omega(\gamma) = 0, \pm 1, \pm 2, \ldots$$

the "quantum degeneracy" of a given species of cycle / particle;
non-zero if and only if such a cycle exists in the geometric sense

Kontsevich-Soibelman conjecture : a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle V_{\gamma + \gamma'} \qquad \langle \gamma, \gamma' \rangle = 0, \pm 1, \pm 2, \ldots$$

$$\Omega(\gamma) = 0, \pm 1, \pm 2, \ldots$$

the "quantum degeneracy" of a given species of cycle / particle;
non-zero if and only if such a cycle exists in the geometric sense

mathematicians say, from 6d viewpoint

intersection number between the two cycles

$$\langle \gamma, \gamma' \rangle$$

Euler number of the moduli space of the cycle

$$\Omega(\gamma)$$

physicists say, from 4d viewpoint

Schwinger product

$$\langle \gamma, \gamma' \rangle = \langle (g,e),(g',e') \rangle = g \cdot e' - e \cdot g'$$

2nd helicity trace = the "number" of species of such particles

$$\Omega(\gamma) = -\frac{1}{2}\mathrm{tr}_\gamma \, (-1)^{2J_3}(2J_3)^2$$

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

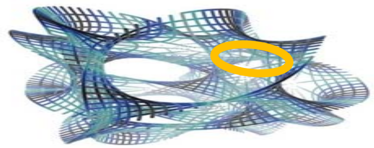
an infinite dimensional representation of
Kontsevich-Soibelman wall-crossing algebra

$$V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle\gamma,\gamma'\rangle} \langle\gamma,\gamma'\rangle V_{\gamma+\gamma'} \qquad \langle\gamma,\gamma'\rangle = 0, \pm 1, \pm 2, \dots$$

$$K_\gamma \equiv \exp\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2}\right)$$

$$K_\gamma \; : \; X_\alpha \; \to \; X_\alpha (1 - \sigma(\gamma) X_\gamma)^{\langle\gamma,\alpha\rangle}$$

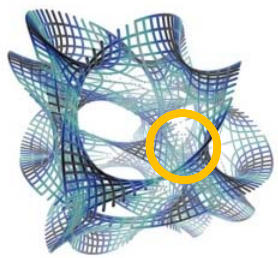
the conjecture : given the left-hand-side, the right-hand-side is entirely determined via the algebraic identity as follows



\+ side

$$\prod_{\gamma} K_{\gamma}^{\Omega^{+}(\gamma)} = \prod_{\gamma'}{}^{\prime} K_{\gamma'}^{\Omega^{-}(\gamma')}$$

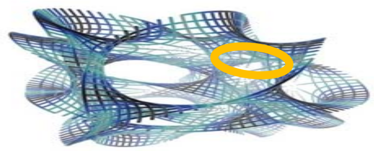
– side

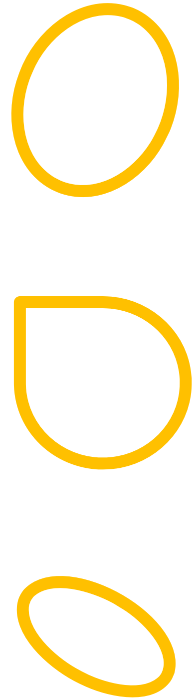


$$K_{\gamma} \equiv \exp\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2}\right)$$

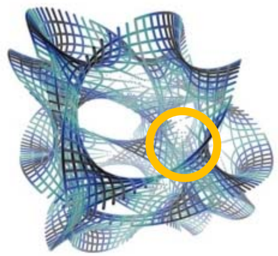
$$\exp(A) \equiv \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

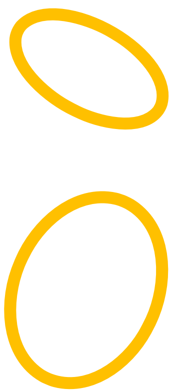
for example, $V_{\gamma_1} V_{\gamma_2} - V_{\gamma_2} V_{\gamma_1} = -V_{\gamma_1+\gamma_2}$



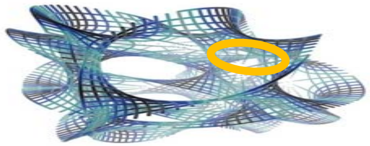


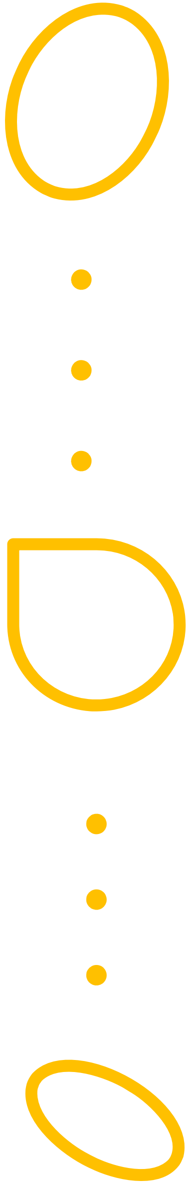
$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1+\gamma_2} K_{\gamma_1}$$



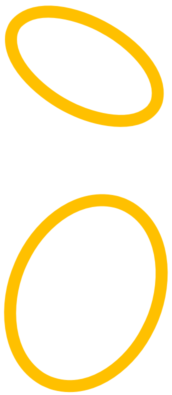


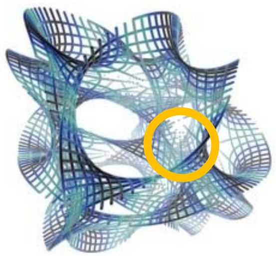
for example, $V_{\gamma_1}V_{\gamma_2} - V_{\gamma_2}V_{\gamma_1} = 2V_{\gamma_1+\gamma_2}$





$$K_{\gamma_1}K_{\gamma_2} = K_{\gamma_2}K_{\gamma_1+2\gamma_2}K_{2\gamma_1+3\gamma_2}\cdots$$
$$\cdots K^{-2}_{\gamma_1+\gamma_2}\cdots K_{3\gamma_1+2\gamma_2}K_{2\gamma_1+\gamma_2}K_{\gamma_1}$$





2000 Stern + P.Y.

wall-crossing formula for simple magnetic charges; weak coupling regime

2002 Denef

quiver dynamics representation of N=2 supergravity BH's

⋮

2008 Kontsevich + Soibelman

⋮

2010/2011 Manschot + Pioline + Sen

general n-particle conjecture for Quantum Mechanics Counting

2011 Lee+P.Y. / Kim+Park+Wang+P.Y.

general n-particle solution to Quantum Mechanics Counting

2011 Sen

Quantum Mechanics Counting = Kontsevich-Soibelman Conjecture

Atiyah-Singer Index Theorem ~ 1963	1975 ~	Bogomolnyi-Prasad-Sommerfeld (BPS)
Calabi-Yau ~ 1978	1977 ~	Supersymmetry
Calibrated Geometry ~ 1982	1983 ~	Superstring Theory
(Harvey & Lawson)	.	
.	1985 ~	Calabi-Yau Compactification
.	1988 ~	Mirror Symmetry
.	.	
Homological Mirror Symmetry ~ 1994	1994 ~	Wall-Crossing Discovered
(Kontsevich)		(Seiberg & Witten)
.	1995 ~	Dirichlet Branes
.	1998 ~	Wall-Crossing is Bound State
.		Dissociation (Lee & P.Y.)
.	.	
Stability & Derived Category ~ 2000	2001 ~	Wall-Crossing for Black Holes
.		(Denef)
.	.	
.	.	
.	.	
.	.	
Wall-Crossing Conjecture~ 2008	2008 ~	Konstevich-Soibelman Explained
(Conjecture by Kontsevich & Soibelman)		(Gaiotto & Moore & Neitzke)
.	.	
.	2011 ~	KS Wall-Crossing proved
.		via Quatum Mechanics
.		Manschot , Pioline & Sen /
		Kim , Park, Wang & P.Y. / Sen

quantum physics count states, gravity makes geometry,
and superstring theory combines quantum & gravity

quantum mechanics "count" geometry via superstring theory

→ quantum mechanical proof of the Kontsevich-Soibelman conjecture
which solves, partially, a 30-year-old geometry problem

geometry as mathematical tools for modern physics

string theory as a physical tool for modern geometry